# Paraconsistency and Program Logics

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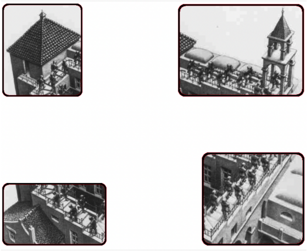
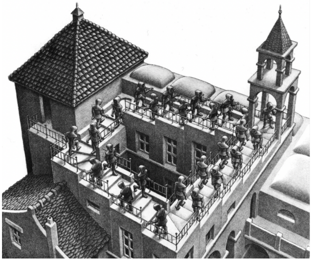
# What's up on the table?

The need to articulate different perspectives on complex information:

local consistency

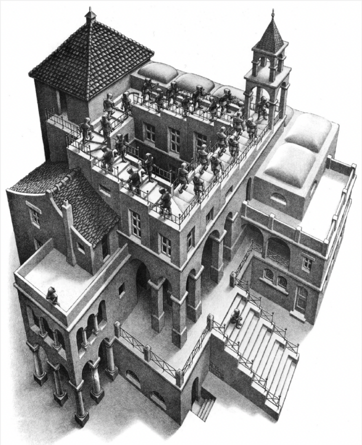


global inconsistency



## ... and why we should care?

Information (reality?) is not what we thought it was ...



M. C. Escher, *Ascending and descending*

# Vagueness and contradiction

In particular, modelling contexts in which bivalent, or even probabilistic reasoning is not enough, entail the need for capturing both

- lack of information (vagueness or uncertainty)
- excess of information (potential inconsistency)

# Vagueness and contradiction

Vagueness is addressed in fuzzy logics,

but potentially contradictory information arises in a number of scenarios. (e.g. knowledge representation, data integration, etc.)

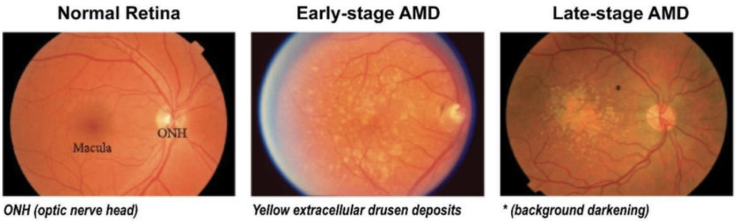
## Two real-world scenarios

- Inference over huge data sets for Age-related Macular Degeneration (AMD) diagnosis
- Identification of success factors in inter-agency partnerships

# Age-related macular degeneration (AMD) diagnosis

A disease of the macula, leading cause of vision loss in people over 55.

- multifactorial disease, with a complex pathophysiology, for which the onset and progression are different, with different risk factors, environmental and genetic, contributing.
- progression not well understood (e.g. different rates/patterns for similar patients and even between the two eyes of the same patient).



# Age-related macular degeneration (AMD) diagnosis

- Highly heterogeneous data collected in large epidemiological studies over long time spans
- Crucial role played by expert assessment: the evidence level assigned to each data factor (e.g. an image) as an enabler for a specific future development of the disease may vary from an expert to another, leading to complex data consolidation processes.
- Thus, potentially contradicting medical judgments cannot be swept under the carpet.

A recently funded proposal supported by a leading eye lab to develop an inference framework able to reason about contradictory, or even inconsistent data in a sound and effective way.

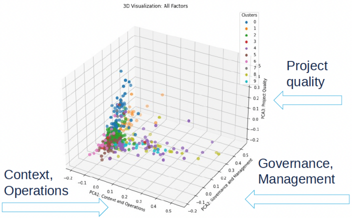
AIBILI association for innovation and biomedical research on light and image

# International inter-agency partnerships

19% - 25% international partnerships are unsuccessful, translating into diminished cooperation impact and public expenditure inefficiency.

- Data collected through text mining over a huge documental fund, resorting to different techniques (XAI, neural nets, etc)
- lead to the identification of 735 classes with opposite-polarity weights from different machine learning models

| Cluster ID | Number of Factors | Topics |
|---|---|---|
| 0 | 67 | Quality-related issues |
| 1 | 32 | Governance and Management frameworks |
| 2 | 343 | Bulky cluster (sub-clusters) |
| 3 | 46 | Mutual commitment |
| 4 | 61 | Country-level capacity |
| 5 | 42 | Staff |
| 6 | 23 | Previous collaboration |
| 7 | 23 | Cost-benefit, cost coverage |
| 8 | 40 | Country-level strategy |
| 9 | 33 | Continued relevance in context |



# Success factors identification in inter-agency partnerships

| factor ID | | factor text | number of semantically similar corpus sentences | |
|---|---|---|---|---|
| 753-factor list | 24-factor list | | positive polarity | negative polarity |
| 18 | 0 | mechanisms for expression of stakeholder viewpoint on the partnership | 45 | 43 |
| 51 | 1 | gender-sensitive equity indicators measuring reach and benefit for disadvantaged groups (PPP) | 54 | 52 |
| 83 | 2 | capacity building | 28 | 37 |
| 152 | 3 | local borrower performance history | 44 | 50 |
| 157 | 4 | staff requirement and capacity assessment | 48 | 52 |
| 162 | 5 | clear, result-oriented, competitive procurement processes, bidding procedures and contracts | OOR | 28 |
| 192 | 6 | partner performance assessment in project completion reports or evaluation reports | 84 | 81 |
| 219 | 7 | project processing and implementation capacity | 31 | OOR |
| 228 | 8 | risk identification, risk management framework at the strategic and project level from the outset | OOR | 40 |
| 263 | 9 | stakeholder involvement | 68 | 72 |
| 300 | 10 | capacity building enabling full participation | 40 | 47 |
| 306 | 11 | focus on outcomes in the partnership strategy | 29 | OOR |
| 308 | 12 | specific guidance on approaches and outcomes in the partnership strategy | 33 | 35 |
| 313 | 13 | cost-benefit analysis | 44 | 32 |
| 379 | 14 | early-on attention to sustainability of project benefits | 57 | 72 |
| 440 | 15 | donors supportive of the aims and operations of the partnership in the local context | 28 | 40 |
| 463 | 16 | capacity building in local partners | OOR | 30 |
| 468 | 17 | quality of project monitoring and evaluation systems design | 61 | 54 |
| 550 | 18 | lack of harmonization of procurement and disbursement procedures | 59 | 69 |
| 557 | 19 | outdated procurement procedures | 45 | 58 |
| 559 | 20 | rigid and diverging procedures for procurement and disbursement | 27 | OOR |
| 602 | 21 | complexity of procurement procedures | OOR | 28 |
| 623 | 22 | lack of stakeholder involvement during initial processes | 31 | OOR |
| 656 | 23 | involvement of stakeholder country in the design of the partnership or of its programmes, for ownership | 32 | 34 |

Joint project to design a inference engine to predict partnership performance from ex-ante documents

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# Summing up: Data is a mined field

- Not only the values and structure of data changes but also the logic under which this information needs to be understood changes as well
- Informational states may exhibit
  - potentially inconsistent (or partially consistent) data,
  - reflecting the diversity of judgements (e.g. from different domain experts).

# Summing up: Data is a mined field

- Moreover, they may be linked
  - positively (witnessing e.g. the existence of a relationship) and
  - negatively (recording whatever prevents such a relationaship)
- Finally, the weights of such transitions are, in most cases, non-complementary, opening an inference arena encompassing both classical, vague, and even (controlled forms of) inconsistent reasoning.

# Paraconsistent Logic

Originally developed in Latin America in 50's, mainly by F. Asenjo and Newton da Costa, accommodates inconsistency in a controlled way, treating inconsistent information as potentially informative.

Paraconsistency is the study of logical systems in which the presence of a contradiction does not imply triviality, therefore avoiding the principle of explosion according to which any statement can be proven from a contradiction.

## Separating

contradiction from deductive triviality
inconsistency from contradiction
consistency from absence of contradiction

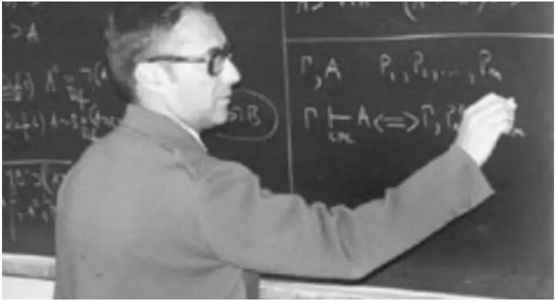
# A brief history of an idea

- 1910 J. Łukasiewicz and N.Vasiliev: denying the law of noncontradiction would lead to still meaningful, although non-Aristotelian logics
- S. Jaśkowski's study of empirical theories including contradictory assumptions
- 1958: Newton da Costa (1929 - 2024) seminal paper

*Nota sôbre o conceito de contradição*.
*Anuario da Sociedade Paranaense de Matematica*, 2 (1), 1958.

paved the way to the remarkable influence of the Brazilian School

# A brief history of an idea



*"I decided to do it the other way round: mathematics with contradictions. Existence in mathematics means anything but the absence of contradiction. Contradictions begin to appear at the edges of mathematics. There are always problems."*

[Newton da Costa, *available from youtube*, 2019]

# A brief history of an idea

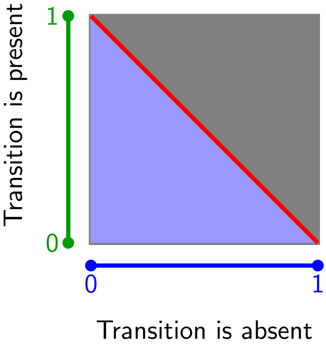
- A main development in the Brazilian School led to the logic of formal inconsistency (LFI) in which the consistency of a sentence is made explicit by a unary operator
  (W. Carnielli, M. E. Coniglio, J. Marcos)
- 1990: *Mathematical Reviews* added a new entry 03B53 entitled Paraconsistent Logic later expanded to Logics admitting inconsistency
- ... and applications pop out: from Philosophy of Science to Mathematics, from Economics, to Quantum Mechanics.
- ... and in Computer Science: AI, databases, semantics of concurrency, quantum computation ...

# A modelling tool

## The approach

- Transition systems with both positive and negative accessibility relations, with non complementary weights:
  - one weighting the possibility of a transition to be present (e.g. the state remaining coherent),
  - the other weighting the possibility of being absent (i.e. becoming unstable)
- used in software modelling and as Kripke frames for a modal logic

# The basic structure



# How is information weighted?

## Underlying semantic structure

- A residuated lattice, i.e. a bounded lattice $\boldsymbol{A} = \langle A, \sqcap, \sqcup, 1, 0 \rangle$ equipped with a monoid $\langle A, \odot, e \rangle$ such that $\odot$ has a right adjoint,

$$a \odot b \leqslant c \;\Leftrightarrow\; b \leqslant a \rightarrow c$$

- st the monoidal operation $\odot$ coincides with meet $\sqcap$
- plus a prelinearity condition:

$$(a \rightarrow b) \sqcup (b \rightarrow a) = 1$$

(iMTL-algebra, after *integral monoidal t-norm based logic*)

# Some examples

$\mathbf{3} = \langle \{\bot, u, \top\}, \wedge_3, \vee_3, \top, \bot, \rightarrow_3 \rangle$

| $\vee_3$ | $\bot$ | $u$ | $\top$ |
|---|---|---|---|
| $\bot$ | $\bot$ | $u$ | $\top$ |
| $u$ | $u$ | $u$ | $\top$ |
| $\top$ | $\top$ | $\top$ | $\top$ |

| $\wedge_3$ | $\bot$ | $u$ | $\top$ |
|---|---|---|---|
| $\bot$ | $\bot$ | $\bot$ | $\bot$ |
| $u$ | $\bot$ | $u$ | $u$ |
| $\top$ | $\bot$ | $u$ | $\top$ |

| $\rightarrow_3$ | $\bot$ | $u$ | $\top$ |
|---|---|---|---|
| $\bot$ | $\top$ | $\top$ | $\top$ |
| $u$ | $\bot$ | $\top$ | $\top$ |
| $\top$ | $\bot$ | $u$ | $\top$ |

$\ddot{\mathbf{G}} = \langle 0..1, min, max, 0, 1, \rightarrow \rangle$ (Gödel)

$$a \rightarrow b = \begin{cases} 1, & \text{if } a \leqslant b \\ b, & \text{otherwise} \end{cases}$$

# How is information weighted?

... endowed with a metric $d : A \times A \longrightarrow \mathbb{R}^+$

## Examples

$\underline{2}$ and $\underline{3}$

| $\underline{2}$ | $\underline{3}$ |
|---|---|
| $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$ | see table below |

| $d$ | $\bot$ | $u$ | $\top$ |
|---|---|---|---|
| $\bot$ | 0 | 1 | 2 |
| $u$ | 1 | 0 | 1 |
| $\top$ | 2 | 1 | 0 |

$\ddot{\mathbf{G}}$ (Gödel)

$$\ddot{\mathbf{G}} = \langle [0,1], min, max, 0, 1, \rightarrow, d \rangle$$

where $d(x,y) = y - x$

# Computing with pairs of weights

A twisted structure, obtained through the direct product of $\boldsymbol{A}$ and its order dual:

$$\boldsymbol{A}^2 = \langle A \times A, \bar{\sqcap}, \bar{\sqcup}, \Rightarrow, /\!\!/ \rangle$$

- $(a, b) \,\bar{\sqcap}\, (c, d) = (a \sqcap c, b \sqcup d)$
- $(a, b) \,\bar{\sqcup}\, (c, d) = (a \sqcup c, b \sqcap d)$
- $(a, b) \Rightarrow (c, d) = ((a \rightarrow c) \sqcap (d \rightarrow b), a \sqcap d)$
- $/\!\!/(a, b) = (b, a)$

with

$$(a, b) \preccurlyeq (c, d) \text{ iff } a \leqslant c \text{ and } b \geqslant d$$

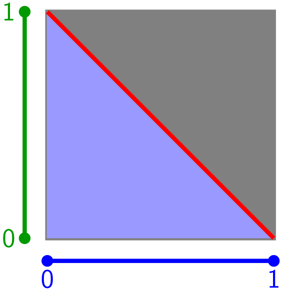
... and a metric $D((a, b), (c, d)) = \sqrt{d(a, c)^2 + d(b, d)^2}$

# Computing with pairs of weights

(paraconsistent): $\Delta_P = \{(a,b) | D((a,b),(1,1)) \leqslant D((a,b),(0,0))\}$

(consistent): $\Delta_C = \{(a,b) | D((a,b),(0,0)) \leqslant D((a,b),(1,1))\}$

(strictly consistent): $\Delta = \Delta_P \cap \Delta_C$



# Computing with pairs of weights

## Alert: Adjunction $\sqcap \dashv \rightarrow$ does not lift

i.e. $(a,b) \sqcap\!\!\sqcap (c,d) \preccurlyeq (e,f)$ iff $(c,d) \preccurlyeq (a,b) \Rightarrow (e,f)$ fails:

e.g. for $(a,b) = (0.8, 0.4)$, $(c,d) = (0.5, 0.2)$ and $(e,f) = (0.6, 0.3)$,

$$(a,b) \sqcap\!\!\sqcap (c,d) = (min\{0.8, 0.5\}, max\{0.4, 0.2\}) = (0.5, 0.4)$$
$$(a,b) \Rightarrow (e,f) = (min(0.8 \rightarrow 0.6, 0.3 \rightarrow 0.4), min\{0.8, 0.3\}) = (0.6, 0.3)$$

Thus,

$$(0.5, 0.4) \preccurlyeq (0.6, 0.3) \text{ but } (0.5, 0.2) \not\preccurlyeq (0.6, 0.3)$$

but is recovered replacing $\sqcap\!\!\sqcap$ by

$$(a,b) \otimes (c,d) = (a \sqcap c, (a \rightarrow d) \sqcap (c \rightarrow b))$$

$$\boxed{(a,b) \otimes (c,d) \preccurlyeq (e,f) \text{ iff } (c,d) \preccurlyeq (a,b) \Rightarrow (e,f)}$$

# Some properties

- The twist construction brings an involution
- and keeps the complete, universally distributive lattice structure, as the product of two such lattices

## Some properties

$$\sslash\left(\sslash(a,b) \Sqcap \sslash(c,d)\right) = (a,b) \Sqcup (c,d)$$

$$\sslash\left(\bigsqcap_{i=1}^{n}(a_i,b_i)\right) = \bigsqcup_{i=1}^{n} \sslash(a_i,b_i)$$

$$\sslash\big((a,b) \Rightarrow \sslash(c,d)\big) = (a,b) \otimes (c,d)$$

# Some properties

...

if $(a',b') \preccurlyeq (a,b)$ and $(c,d) \preccurlyeq (c',d')$ then $(a,b)[\Rightarrow|\otimes](c,d) \preccurlyeq (a',b') \Rightarrow (c',d')$

$$(a,b) \otimes \Big((c,d) \sqcup\!\!\!\sqcup (e,f)\Big) = \Big((a,b) \otimes (c,d)\Big) \sqcup\!\!\!\sqcup \Big((a,b) \otimes (e,f)\Big)$$

$$(a,b) \otimes \Big((c,d) \sqcap\!\!\!\sqcap (e,f)\Big) \preccurlyeq \Big((a,b) \otimes (c,d)\Big) \sqcap\!\!\!\sqcap \Big((a,b) \otimes (e,f)\Big)$$

$$(a,b) \otimes \Big((c,d) \Rightarrow (e,f)\Big) \preccurlyeq (c,d) \Rightarrow \Big((a,b) \otimes (e,f)\Big)$$

$$(a,b) \Rightarrow \Big((c,d) \Rightarrow (e,f)\Big) \preccurlyeq \Big((a,b) \otimes (c,d)\Big) \Rightarrow (e,f)$$

$$(a,b) \otimes \Big((c,d) \Rightarrow (e,f)\Big) \preccurlyeq \Big((a,b) \Rightarrow (c,d)\Big) \Rightarrow (e,f)$$

$$(a,b) \Rightarrow \Big((c,d) \Rightarrow (e,f)\Big) \preccurlyeq (c,d) \Rightarrow \Big((a,b) \Rightarrow (e,f)\Big)$$

...

# Paraconsistent Labelled Transition Systems

## The approach

- Introduce labels from a set of identifiers *Act*, and an initial state
- Define morphism to organise PLTS over $\boldsymbol{A}$ into a category
- Derive an algebra, to get new PLTS from old, from the underlying categorical structure
  (... possibly leading to a language and a dynamic logic)
- Develop a multimodal logic (*à la* Hennessy-Milner) for PLTS

# Paraconsistent Labelled Transition Systems

A PLTS over a iMTL-algebra $\boldsymbol{A}$, and a set of atomic actions $Act$ is a structure

$$T = \langle W, i, R \rangle$$

where,

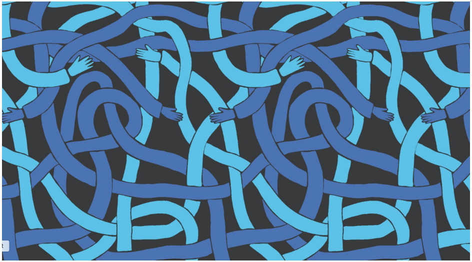
- $W$ is a non-empty set of states
- $i \in W$ is the initial state
- $R = (R_a \colon W \times W \longrightarrow A \times A)_{a \in Act}$ is an $Act$-indexed family of functions

$$R_a(w_1, w_2) = (\alpha, \beta)$$

with $\alpha$ weighting evidence of the transition through $a$ and $\beta$ its absence.

# Where can we go from here?

- Modelling (Programming): requires a notion of morphism to discuss compositionality
- Reasoning (Verification): requires extension to a Kripke structure, a notion of (bi)simulation and a logic



# Morphism

$(h, \lambda) : T \rightarrow T'$

- $h : W \rightarrow W'$
- $\lambda : Act \rightarrow_{\perp} Act'$

such that

- $h(i) = i'$ and
- for any $a \in Act$,

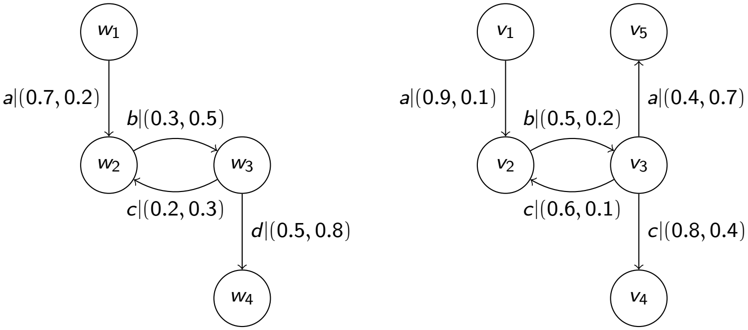
$$R_a(w, w') \preccurlyeq R'^{\perp}_{\lambda(a)}(h(w), h(w'))$$

where $R^{\perp} = R \cup R_{\perp}$ with $R_{\perp}(w, w) = (1, 0)$ for any state $w \in W$

# Morphism

## Example

$$(h, id) \text{ with } h = \{w_1 \mapsto v_1, w_2 \mapsto v_2, w_3 \mapsto v_3, w_4 \mapsto v_4\}$$



# Kripke Structures & (Bi)simulation

A PLTS $T = \langle W, i, R \rangle$ over a iMTL-algebra $\boldsymbol{A}$, and a set of atomic actions $Act$, generates a Kripke structure

$$\langle W, R, V \rangle$$

where

$$V : W \times Prop \rightarrow A \times A$$

is a valuation function over a set $Prop$ of proposition symbols.

# Kripke Structures & (Bi)simulation

## The crisp case

$S \subseteq W \times W'$ is a simulation if, for all $p \in Prop$, $a \in Act$,

if $wSw'$ then

- $V(w, p) \preccurlyeq V'(w', p)$
- If $R_a(w, v) = (\alpha, \beta)$ then,

$$\exists_{w' \in W'}.\ R_a(w', v') = (\delta, \gamma) \text{ and } w'Sv' \text{ and } (\alpha, \beta) \preccurlyeq (\delta, \gamma)$$

# Kripke Structures & (Bi)simulation

## The crisp case

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- If $R_a(w,v) = (\alpha, \beta)$ then,

$$\exists_{w' \in W'}.\ R_a(w',v') = (\delta, \gamma) \text{ and } w'Sv' \text{ and } (\alpha,\beta) \preccurlyeq (\delta,\gamma)$$

## The graded case

$G : W \times W' \to A \times A$ is a graded simulation if, for all $p \in Prop$, $a \in Act$,

- $G(w,w') \preccurlyeq (V(w,p) \Rightarrow V'(w',p))$
- $\exists_{v' \in W'} \ (G(w,w') \otimes R_a(w,v)) \preccurlyeq (R'_a(w',v') \otimes G(v,v'))$

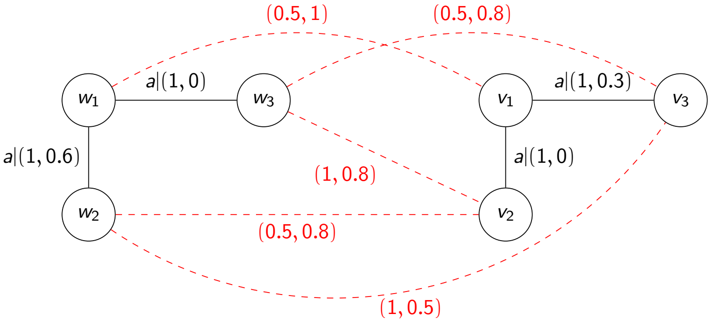
# Kripke Structures & (Bi)simulation

## Graded bisimulation

$G : W \times W' \rightarrow A \times A$ is a graded bisimulation if, for all $p \in Prop$, $a \in Act$,

- $G(w, w') \preccurlyeq (V(w, p) \Leftrightarrow V'(w', p))$
- $\exists_{v' \in W'} \ (G(w, w') \otimes R_a(w, v)) \preccurlyeq (R'_a(w', v') \otimes G(v, v'))$
- $\exists_{v \in W} \ (G(w, w') \otimes R'_a(w', v')) \preccurlyeq (R_a(w, v) \otimes G(v, v'))$

# Kripke Structures & (Bi)simulation




where

$V(w_1, p) = V'(v_1, p) = 1, 1)$

$\mathrm{V}(\mathrm{w}_2, p) = V'(v_3, p) = (1, 0.5)$

$\mathrm{V}(\mathrm{w}_3, p) = V'(v_2, p) = (0.8, 0.8)$

# Kripke Structures & (Bi)simulation

Alert: Crisp (bi)simulations are not necessarily graded

i.e. a crisp (bi)simulation $S$ st $wSw'$ does not entail the existence of a graded one $G$ st $G(w, w') = (1, 0)$.

- Conditions enforcing coincidence are identified for both simulation and bisimulation
- This helps to explain some weird behaviour of graded behavioural relations.

# Kripke Structures & (Bi)simulation

## Example


with $V(w,p) = V'(w',p) = (0.2, 0.2)$.

Relation $G(w,w') = (1,0)$ is not a graded simulation:

$$
\begin{aligned}
(1,0) \not\preccurlyeq (0.2, 0.2) \Rightarrow (0.2, 0.2) \\
\not\preccurlyeq (1, 0.2)
\end{aligned}
$$

# Paraconsistency for the working software engineer

- Modelling (Programming): New PLTS from old

  .. exploring the underlying category $Pt_{\mathbf{A}}$ of PLTS and their morphisms

- Reasoning (Verification): A multimodal logic $PML(\mathbf{A})$

  ... action-indexed modalities interpreted over paraconsistent Kripke structures

# New PLTS from old

## Restriction and relabeling

Let $T$ be a PLTS. Then,

- Restriction: for $\lambda : Act' \rightarrow Act$ an inclusion, $T \upharpoonright \lambda$ is a Cartesian lifting in $Pt_{\mathbf{A}}$, i.e., any other morphism whose action component is $\lambda$ factors through the restriction morphism from $T \upharpoonright \lambda$ to $T$.
- Relabelling: for $\lambda : Act' \rightarrow Act$ total, $T\{\lambda\}$ is a co-Cartesian lifting in $Pt_{\mathbf{A}}$

# New PLTS from old

## New PLTS from old: Parallel Composition

$T_1 \times T_2$ is the categorical product in $Pt_{\boldsymbol{A}}$:

$$T_1 \times T_2 = \langle W_1 \times W_2, (i_1, i_2), R \rangle$$

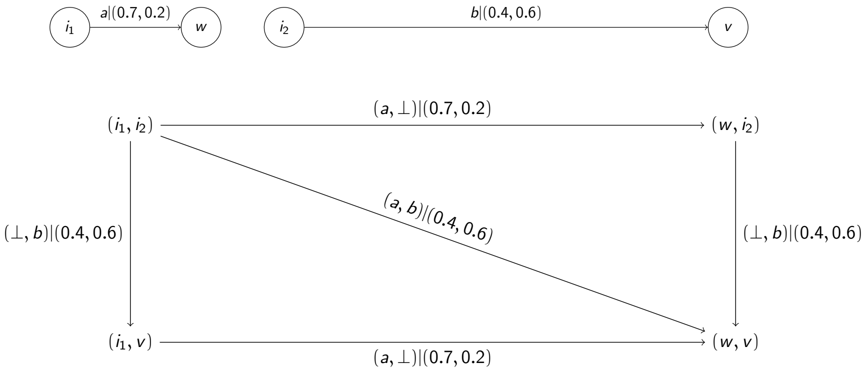
over

$$Act_1 \times_\perp Act_2 = \{(a, \perp) \mid a \in Act_1\} \cup \{(\perp, b) \mid b \in Act_2\} \cup \{(a, b) \mid a \in Act_1, b \in Act_2\}$$

such that

$$\begin{aligned} R_{(a,b)}((w_1, w_2), (v_1, v_2)) &= (\alpha, \beta) \text{ iff} \\ (R_1)_a^\perp(w_1, v_1) &= (\alpha_1, \beta_1) \text{ and} \\ (R_2)_b^\perp(w_2, v_2) &= (\alpha_2, \beta_2) \text{ and} \\ (\alpha, \beta) &= (\alpha_1, \alpha_2) \sqcap\!\!\sqcap (\beta_1, \beta_2) \end{aligned}$$

# Example



$i_1$
$a|(0.7, 0.2)$
$w$
$i_2$
$b|(0.4, 0.6)$
$v$
$(i_1, i_2)$
$(a, \perp)|(0.7, 0.2)$
$(w, i_2)$
$(\perp, b)|(0.4, 0.6)$
$(a, b)|(0.4, 0.6)$
$(\perp, b)|(0.4, 0.6)$
$(i_1, v)$
$(w, v)$
$(a, \perp)|(0.7, 0.2)$

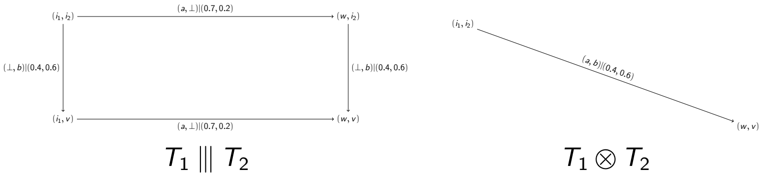
# New PLTS from old

Interleaving: $T_1 \mid\mid\mid T_2 \;=\; (T_1 \times T_2) \restriction \lambda$

with the inclusion $\lambda : \{(a, \bot) \mid a \in Act_1\} \cup \{(\bot, b) \mid b \in Act_2\} \longrightarrow Act_1 \times_\bot Act_2$

Synchronous product: $T_1 \otimes T_2 \;=\; (T_1 \times T_2) \restriction \lambda$

taking $\{(a, b) \mid a \in Act_1$ and $b \in Act_2\}$ as the domain of $\lambda$



# New PLTS from old

## Choice

$T_1 + T_2$ is the categorical coproduct in $Pt_{\boldsymbol{A}}$:

$$T_1 + T_2 = \langle W, (i_1, i_2), R \rangle$$

over $Act = Act_1 \cup Act_2$, where

- $W = (W_1 \times \{i_2\}) \cup (\{i_1\} \times W_2)$
- $R_a((w_1, w_2), (v_1, v_2)) = (\alpha, \beta)$ iff $(R_1)_a(w_1, v_1) = (\alpha, \beta)$ or $(R_2)_a(w_2, v_2) = (\alpha, \beta)$

$$(i_1, v) \xleftarrow{b|(0.4, 0.6)} (i_1, i_2) \xrightarrow{a|(0.7, 0.2)} (w, i_2)$$

# New PLTS from old

## Other Operators

- Sequential composition as prefixing
- Functorial extension of operations from the underlying iMTL-algebra (e.g. to operate on weights)

... leading to a sort of (paraconsistent) process algebra

# $PML(\boldsymbol{A})$: A multimodal logic for PLTS

Given an iMTL-algebra $\boldsymbol{A}$, a set of proposition symbols $Prop$ and a set of action symbols $Act$:

$$\varphi \coloneqq \bot \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid [a]\varphi$$

where $p \in Prop$ and $a \in Act$.

Abbreviations:

- $\top = \neg\bot$
- $\varphi \vee \varphi' = \neg(\neg\varphi \wedge \neg\varphi')$
- $\varphi \rhd \varphi' = \neg\varphi \vee \varphi'$
- $\varphi \lhd\!\rhd \varphi' = \varphi \rhd \varphi' \wedge \varphi' \rhd \varphi$
- $\langle a \rangle\varphi = \neg[a]\neg\varphi$

# The logic $PML(\mathbf{A})$

Satisfaction: $\models : M \times Sen(Prop, Act) \rightarrow A \times A$

$$(M \models \varphi) = \bigsqcap_{w \in W} (M, w \models \varphi)$$

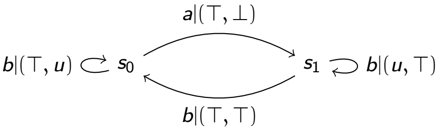
- $(M, w \models \bot) = (0, 1)$
- $(M, w \models p) = V(w, p)$
- $(M, w \models \neg\varphi) = /\!\!/(M, w \models \varphi)$
- $(M, w \models \varphi \wedge \varphi') = (M, w \models \varphi) \sqcap (M, w \models \varphi')$
- $(M, w \models [a]\varphi) = \bigsqcap_{v \in W} (R_a(w, v) \Rightarrow M, v \models \varphi)$

# The logic $PML(\boldsymbol{A})$

## Deriving $\langle a\rangle\varphi$ (the role of the adjunction)

$$
\begin{aligned}
(M, w \models \langle a\rangle\varphi) &= (M, w \models \neg[a]\neg\varphi) \\
&= /\!\!/ \, (M, w \models [a]\neg\varphi) \\
&= /\!\!/ \left( \bigsqcap_{v\in W} \left( R_a(w,v) \Rightarrow (M, v \models \neg\varphi) \right)\right) \\
&= /\!\!/ \left( \bigsqcap_{v\in W} \left( R_a(w,v) \Rightarrow /\!\!/ (M, v \models \varphi) \right)\right) \\
&= \bigsqcup_{v\in W} /\!\!/ \left( R_a(w,v) \Rightarrow /\!\!/ (M, v \models \varphi) \right) \\
&= \bigsqcup_{v\in W} \left( R_a(w,v) \otimes (M, v \models \varphi) \right)
\end{aligned}
$$

# Example over 3




| $V$ | $p$ | $q$ | $r$ |
|---|---|---|---|
| $s_0$ | $(\top, \top)$ | $(\bot, u)$ | $(u, u)$ |
| $s_1$ | $(\bot, u)$ | $(\bot, \bot)$ | $(u, u)$ |

$$M, s_1 \models r \rhd (p \vee q)$$

$$\begin{aligned} M, s_0 \models r \rhd (p \vee q) &= /\!\!/ \,(M, s_0 \models r) \uplus (M, s_0 \models (p \vee q)) \\ &= /\!\!/ \, V(s_0, r) \uplus (V(s_0, p) \uplus V(s_0, q)) \\ &= (u, u) \uplus (\top, \top) \uplus (\bot, u) \\ &= (u \vee_3 \top \vee_3 \bot, u \wedge_3 \top \wedge_3 u) \\ &= (\top, u) \end{aligned}$$

# Example over 3

$$M, s_1 \models \langle b \rangle p$$

$$\begin{aligned}
M, s_1 \models \langle b \rangle p &= M, s_1 \models /\!\!/ [b] /\!\!/ \, p \\
&= /\!\!/ \left( \bigsqcap_{s \in W} \widehat{R}_b(s_1, s) \Rightarrow /\!\!/ (M, s \models p) \right) \\
&= /\!\!/ \left( \left( R_b(s_1, s_0) \Rightarrow /\!\!/ V(s_0, p) \right) \sqcap \left( R_b(s_1, s_1) \Rightarrow /\!\!/ V(s_1, p) \right) \right) \\
&= /\!\!/ \left( \left( (\top, \top) \Rightarrow (\top, \top) \right) \sqcap \left( (u, \top) \Rightarrow (u, \bot) \right) \right) \\
&= /\!\!/ \, (\top \wedge_3 \top, \top \vee_3 \bot) \\
&= (\top, \top)
\end{aligned}$$

# Bisimulation and Modal Invariance

- $S$ crisp bisimulation:

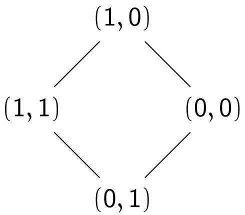
$$\langle w, w' \rangle \in S \text{ entails } (M, w \models \varphi) = (M', w' \models \varphi)$$

- $G$ graded bsimulation:

$$G(w, w') \preccurlyeq \Big( (M, w \models \varphi) \Leftrightarrow (M', w' \models \varphi) \Big)$$

# Paraconsistent Structures and their Logics

- Generalising Belnap-Dunn *FOUR*
- to PLTS parametric on a iMTL-algrebra, and corresponding twisted strucutre,
- combinators and a multimodal logic




- Cruz, Madeira & Barbosa: A logic for paraconsistent transition systems
  *Non-Classical Logics: Theory and Applications*, 2022.
- Cruz, Madeira & Barbosa: Paraconsistent transition systems
  LSFA'22 (Logical and Semantic Frameworks with Applications), 2022.
- Barbosa & Madeira: Capturing qubit decoherence through paraconsistent transition systems. Engineering of Quantum Programming , IEEE 2023
- ⋆⋆ Cunha, Madeira & Barbosa: Paraconsistent transition structures: compositional principles and a modal logic. *Math. Struc. Comp. Sci.*, Elsevier (in print) 2025

# Structured Specification and Development

- Instantiation of Sannella and Tarlecki's stepwise implementation for the structured specification of PLTS from their abstract design down to the concrete implementation stage.
- Structured specification logic *à la* CASL
- Paraconsistent institution parametric on a twisted structure enriched with regular modalities


- Cunha, Madeira & Barbosa: Stepwise Development of Paraconsistent Processes
  TASE'23 (Theoretical Aspects of Software Engineering), 2023.
- Cunha, Madeira & Barbosa: Structured specification of paraconsistent transition systems
  FSEN'23 (Fundamentals of Software Engineering), 2023.
- ⋆⋆ Cunha, Madeira & Barbosa: Specification of paraconsistent transition systems, revisited. *Sci Comp Porgramming*, , 240, Elsevier, 2024

# PKAT: The algebraic counterpart

- Study paraconsistent KAT to reason about uncertain or inconsistent computations in a (quasi)-equational way
- made concrete in an algebra of paraconsistent relations


- Cunha, Madeira & Barbosa: Paraconsistent relations as a variant of Kleene algebras
  LSFA'24 (in print), 2024.

# Future/current research

A lot remains to be done ...

- Simulator and model finder
- Classical and graded soundness
- Extensions to reactive transition structures
- Coalgebraic rendering of PLTS: observational equivalences and modal logics for free ...
- Dynamic logics for paraconsistent programming, applied to NISQ (noisy intermediate-scale quantum) programs

# Dynamic logics for NISQ quantum programs

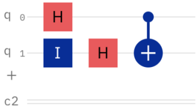
## Context

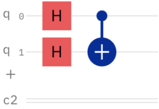
- LQP (Smets & Baltag, 2006)
  – the quantum 'version' of propositional dynamic logic
- PLQP & Company (Smets & Baltag, 2014)
  – with a probability modality to capture the success of a test, allowing for going beyond 'qualitative' properties.

aiming at

- handling hybrid (quantum/classical) programs
- ... and dealing with circuit decoherence profilles and noisy gates

# Dynamic logics for NISQ quantum programs




$q[0]; q[1]$
$\downarrow (H \otimes I, 1, 0)$
$q[0]; q[1]$
$\downarrow (I \otimes H, 0.8, 0.5)$
$q[0]; q[1]$
$\downarrow (CNOT, 0.4, 0.9)$
$q[0]; q[1]$

$q[0]; q[1]$
$\downarrow (H \otimes H, 1, 0)$
$q[0]; q[1]$
$\downarrow (CNOT, 0.6, 0.7)$
$q[0]; q[1]$

Exercise with gates decoherence profiles

... leading, at each computational stage, to throughput/decoherence measures which are not 'complementary' in any sense...

# Concluding: Paraconsistency is everywhere ...

When the hills are flat,
The rivers are all dry.

When it thunders in winter,
When it snows in summer.
When heaven and earth mingle,

Not till then will I part.

Yuefu poems, Han dynasty
(206 BC - 220 AD)